MODULAR REPRESENTATIONS OF THE ORTHO-SYMPLECTIC SUPERGROUPS

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Abstract. A Chevalley type integral basis for the ortho-symplectic Lie superalgebra is constructed. The simple modules of the ortho-symplectic supergroup over an algebraically closed field of prime characteristic not equal to 2 are classified, where a key combinatorial ingredient comes from the Mullineux conjecture on modular representations of the symmetric group. A Steinberg tensor product theorem for the ortho-symplectic supergroup is also obtained.

1. Introduction

1.1. Lie superalgebras, supergroups, and their representation theory over the field of complex numbers $\mathbb{C}$ have been studied extensively in literature since the classification of finite-dimensional complex simple Lie superalgebras by Kac [9]. More on supergroups and supergeometry over $\mathbb{C}$ can be found in the book of Manin [12]. In recent years, the modular representations of algebraic supergroups $GL(n|m)$ and $Q(n)$ over an algebraically closed field k of characteristic $p \neq 2$ have been initiated by Brundan, Kleshchev and Kujawa [2, 3, 4, 10].

The modular representation theory of *super*groups not only is of intrinsic interest in its own right (with the rich classical results in representations of algebraic groups [8] in mind), but also has found remarkable applications to classical mathematics: the classification of simple modules of the spin symmetric group over k in [2] using $Q(n)$, and a new conceptual proof in [4] using $GL(n|m)$ of the celebrated Mullineux conjecture [13] which describes the correspondence of simple modules of the symmetric group S_n over k upon tensoring with the 1-dimensional sign module. The classification of the simple $Q(n)$-modules was also nontrivial [3], in contrast to the algebraic group setup (cf. Jantzen [8]).

1.2. The goal of this paper is to initiate the study of modular representations of the ortho-symplectic supergroup $SpO(2n|\ell)$ over an algebraically closed field k of characteristic $p > 2$. We construct an integral basis (called Chevalley basis as usual) for Lie superalgebra $spo(2n|\ell)$ and classify the simple modules of the algebraic supergroup $SpO(2n|\ell)$ for every n and ℓ.

Recall that the ortho-symplectic Lie superalgebra $spo(2n|\ell)$, which contains $sp(2n) \oplus so(\ell)$ as its even Lie subalgebra, provides the other infinite-series classical superalgebras besides type A in the list of [9]. Let us exclude the classical Lie algebras by assuming $n \geq 1$ and $\ell \geq 1$ here. The infinite series $spo(2n|\ell)$ is further divided into four infinite families by root systems: the series $B(0, n)$ corresponding

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to $\ell = 1$; the series $C(n)$ corresponding to $\ell = 2$; the series $B(m, n)$ for $\ell = 2m+1$ and the series $D(m, n)$ for $\ell = 2m$, where $m \geq 1$. Already over $\mathbb{C}$, the finite-dimensional representation theory of $B(m, n)$ and $D(m, n)$ is very challenging and remains to be better understood (see Serganova [15]). The nontrivial classification of finite-dimensional simple $spo(2n|\ell)$-modules was obtained in [9].

1.3. In Section 2, we establish by a simple and uniform approach an equivalence of categories of rational G-modules and of locally finite $(Dist(G), T)$-modules under some natural assumptions on an algebraic supergroup G, where T is a maximal torus of G and $Dist(G)$ denotes the superalgebra of distributions of G. The verification of the assumptions for the equivalence of categories theorem for $SpO(2n|\ell)$ will follow from results in Section 3. These assumptions can be easily checked for supergroups $GL(n|m)$ and $Q(n)$. Such an equivalence of categories were earlier established in [3] for $Q(n)$ and later in [4] for $GL(n|m)$ by obtaining in an elementary yet ad hoc case-by-case method an isomorphism between a restricted dual of the superalgebra of distributions and the coordinate superalgebra of the supergroup.

We then introduce in Section 3 a Chevalley basis for the Lie superalgebra $spo(2n|\ell)$ for any n, ℓ. While our constructions of the Chevalley basis are carried out explicitly case by case, we observe a uniform phenomenon quite similar to the characterization of Chevalley basis for simple Lie algebras (cf. Steinberg [17, Theorem 1]). The Chevalley basis further leads to a Kostant (integral) basis for the superalgebra of distributions $Dist(SpO(2n|\ell))$. In establishing these integrality statements, we encounter a new phenomenon where twice of an (odd) root could sometimes be an (even) root for $spo(2n|\ell)$.

In Section 4, we establish an analog of the Steinberg tensor product theorem for the supergroup $G = SpO(2n|\ell)$. Once we have the equivalence of categories in place (see Section 2), our approach is quite parallel to [3, 4, 10], which in turn followed a strategy in the algebraic group setup (cf. Cline-Parshall-Scott [5] and [8]). The equivalence of categories in Section 2 allows us to study G-modules using the highest weight module theory of $Dist(G)$. However a new major difficulty arises (when $\ell \geq 3$): not every simple highest weight $Dist(G)$-module $L(\lambda)$ for $\lambda \in X^+(T)$ is finite-dimensional, where $X^+(T)$ denotes the set of dominant integral weights for the underlying even subgroup of G.

In Section 5, we determine completely the subset $X^\dagger(T)$ of $X^+(T)$ which parameterize the simple $SpO(2n|\ell)$-modules. Note that the subset $X^\dagger(T)$ differs from $X^+(T)$ already in characteristic zero (cf. [9]). Remarkably, a key combinatorial ingredient in Mullineux conjecture singles out the subset $X^\dagger(T)$, which depends on the characteristic p of the ground field in general. We refer to the Introduction of [4] for more references and history on the solution of Mullineux conjecture by Kleshchev and others (also cf. [6, 18]).

The main tool in the proof of our classification is the method of odd reflections which has been used over $\mathbb{C}$ by Serganova *et al* (cf. [11, 14, 15]) and then also used by Brundan-Kujawa [4] for $GL(n|m)$ in positive characteristic. To a large extent, our proof was inspired by the Brundan-Kujawa classification of simple *polynomial* $GL(n|m)$-modules, i.e. the simple subquotients appearing in various tensor powers

of the natural $GL(n|m)$-module. It is a remarkable and puzzling coincidence that the simple $SpO(2n|2m+1)$-modules and simple polynomial $GL(n|m)$-modules are classified by an identical set of weights. Our classification which is valid over $\mathbb{C}$ can be shown to be equivalent to the classification in [9] over $\mathbb{C}$ using Dynkin labels where a totally different argument was sketched.

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Convention. The terminology of ideals, subalgebras, modules etc. of a Lie superalgebra instead of superideals, subsuperalgebras, supermodules, etc. is adopted in this paper.

2. An equivalence of module categories

2.1. **Algebraic supergroups.** We first briefly recall the generalities on algebraic supergroups, following [3, Section 2], (which is in turn a generalization of the approach by Demazure-Gabriel and Jantzen [8]), also cf. [12].

Let k be a fixed algebraically closed field of characteristic $p \neq 2$. All objects in this paper will be defined over k unless otherwise specified. Let $A = A_{\bar{0}} + A_{\bar{1}}$ be a commutative superalgebra (i.e. $\mathbb{Z}_2$-graded algebra) over k, i.e. $ab = (-1)^{|a||b|}ba$ for all homogeneous elements $a, b \in A$ of degree $|a|, |b| \in \mathbb{Z}_2$. In the sequel, we assume that all formulas are defined via the homogeneous elements and extended by linearity. An element in $A_{\bar{0}}$ (resp. $A_{\bar{1}}$) is called *even* (resp. *odd*). From the supercommutativity it follows that $a^2 = 0$ for all $a \in A_{\bar{1}}$. We will denote by $\mathfrak{salg}$ the category of commutative superalgebras over k and even homomorphisms. A fundamental object in $\mathfrak{salg}$ is the free commutative superalgebra $k[x_1, \dots, x_n; \xi_1, \dots, \xi_m]$ in even generators x_i and odd generators ξ_j.

An affine superscheme X will be identified with its associated functor in the category of superschemes

$$\mathrm{Hom}(Spec(-), X) : \mathfrak{salg} \longrightarrow \mathfrak{sets}.$$

The affine superscheme $\mathbb{A}^{n|m} := Spec\, k[x_1, \dots, x_n; \xi_1, \dots, \xi_m]$ as a functor sends a commutative superalgebra A to $\mathbb{A}^{n|m}(A) = A_{\bar{0}}^n + A_{\bar{1}}^m$. For an affine superscheme X, its coordinate superalgebra $k[X]$ is the superalgebra $\mathrm{Mor}(X, \mathbb{A}^{1|1})$ of all natural transformations from the functor X to $\mathbb{A}^{1|1}$. One has $X = \mathrm{Hom}_{\mathfrak{salg}}(k[X], -)$.

An affine algebraic supergroup G is a functor from the category $\mathfrak{salg}$ to the category of groups, which associates to a commutative superalgebra A a group $G(A)$ functorially, and whose coordinator algebra $k[G]$ is finitely generated. For an algebraic supergroup G, $k[G]$ admits a canonical structure of Hopf superalgebra, with comultiplication $\Delta : k[G] \to k[G] \otimes k[G]$, the antipode $S : k[G] \to k[G]$, and the counit $\varepsilon : k[G] \to k$. Set

$$\mathcal{J} := \ker(\varepsilon).$$

A closed subgroup of G is an affine supergroup scheme whose coordinate algebra is a quotient of $k[G]$ by a Hopf ideal I. In particular, the underlying purely even group of G, denoted by $G_{\rm ev}$, corresponds to the Hopf ideal $k[G]k[G]_{\bar{1}}$. That is, $k[G_{\rm ev}] \cong k[G]/k[G]k[G]_{\bar{1}}$.

In the remainder of the paper, an affine algebraic supergroup will simply be referred to as a supergroup.

2.2. **Superalgebra of distributions.** Let G be a supergroup. The superspace of distributions (at the identity $e \in G$) is

$$\mathrm{Dist}(G) := \cup_{n\geq 0}\mathrm{Dist}_n(G)$$

where $\mathrm{Dist}_n(G) := \{X \in k[G]^* \mid X(\mathfrak{J}^{n+1}) = 0\} \cong (k[G]/\mathfrak{J}^{n+1})^*$.

The space $\mathrm{Dist}(G)$ is a cocommutative Hopf superalgebra whose multiplication $*$ is dual to the comultiplication Δ of $k[G]$ just as in the ordinary case (cf. [3] and [8]). Furthermore, $\mathrm{Dist}(G)$ is a filtered superalgebra given by:

$$k \subset \mathrm{Dist}_1(G) \subset \cdots \subset \mathrm{Dist}_r(G) \subset \mathrm{Dist}_{r+1}(G) \subset \cdots.$$

For $f_1, \cdots, f_n \in \mathfrak{J}$ and $n \in \mathbb{N}$, we have

$$\Delta(f_1 \cdots f_n) \in \prod_{i=1}^{n}(1 \otimes f_i + f_i \otimes 1) + \sum_{r=1}^{n} \mathfrak{J}^r \otimes \mathfrak{J}^{n+1-r}.$$

It follows that for $X \in \mathrm{Dist}_s(G)$ and $Y \in \mathrm{Dist}_t(G)$

$$[X,Y] := X * Y - (-1)^{|X||Y|} Y * X \in \mathrm{Dist}_{s+t-1}(G).$$

Hence, the tangent space at the identity

$$T_e(G) := \{X \in \mathrm{Dist}_1(G) \mid X(1) = 0\} \cong (\mathfrak{J}/\mathfrak{J}^2)^*$$

carries a Lie superalgebra structure; it is called the Lie superalgebra of G and will be denoted by $\mathrm{Lie}(G)$.

2.3. **The restricted structure on $Lie(G)$.**

Definition 2.1. A Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} + \mathfrak{g}_{\bar{1}}$ is called a *restricted* Lie superalgebra (or p-Lie superalgebra), if the following conditions are satisfied:

(a) $\mathfrak{g}_{\bar{0}}$ is a restricted Lie algebra with p-mapping $[p] : \mathfrak{g}_{\bar{0}} \to \mathfrak{g}_{\bar{0}}$ [7, Chap. 4].
(b) $\mathfrak{g}_{\bar{1}}$ is a restricted $\mathfrak{g}_{\bar{0}}$-module via the adjoint action, i.e. $\mathrm{ad}(X^{[p]})(X_1) = \mathrm{ad}(X)^p(X_1)$, for $X \in \mathfrak{g}_{\bar{0}}, X_1 \in \mathfrak{g}_{\bar{1}}$.

Let G be a supergroup. The canonical map $\pi : k[G] \to k[G_{\rm ev}] = k[G]/k[G]k[G]_{\bar{1}}$ sends $\mathfrak{J}$ to the kernel $\mathfrak{J}_{\rm ev}$ of $\varepsilon_{\rm ev} : k[G_{\rm ev}] \to k$ and $\pi(\mathfrak{J}^i) \subset \mathfrak{J}_{\rm ev}^i$ for $i \geq 1$. This induces an injective algebra homomorphism $\pi^* : \mathrm{Dist}(G_{\rm ev}) \to \mathrm{Dist}(G)$. π also induces an isomorphism of vector spaces from $(\mathfrak{J}/\mathfrak{J}^2)_{\bar{0}}$ to $\mathfrak{J}_{\rm ev}/\mathfrak{J}_{\rm ev}^2$, and both spaces are isomorphic to the space $\mathfrak{J}_{\bar{0}}/(\mathfrak{J}_{\bar{0}}^2 + k[G]_{\bar{1}}^2)$. Thus, we have the following.

Lemma 2.2. *The superalgebra homomorphism π^* induces an isomorphism of Lie algebras from $Lie(G_{\rm ev})$ onto $Lie(G)_{\bar{0}} = Lie(G) \cap Dist(G)_{\bar{0}}$.*

Proposition 2.3. *Let G be a supergroup. Then, Lie(G) is a restricted Lie superalgebra with the p-mapping: $X \mapsto X^{[p]}$ for $X \in Lie(G)_{\bar{0}}$, where $X^{[p]} := \overbrace{X * \cdots * X}^{p}$ is defined in Dist(G). Moreover, the restricted structure on Lie(G_{ev}) as a subalgebra of Lie(G) coincides with the one induced as Lie algebra of the algebraic group G_{ev}.*

Proof. Since G_{ev} is an algebraic group, $\mathrm{Lie}(G_{\mathrm{ev}})$ is a restricted Lie algebra with the p-mapping given by the p-th power in $\mathrm{Dist}(G_{\mathrm{ev}})$ (cf. [1, 8]). The compatibility of the restricted structures on $\mathrm{Lie}(G)$ and on $\mathrm{Lie}(G_{\mathrm{ev}})$ now follows from Lemma 2.2 since $\pi^* : \mathrm{Dist}(G_{\mathrm{ev}}) \to \mathrm{Dist}(G)$ is an algebra homomorphism.

For $X_0 \in \mathrm{Lie}(G)_{\bar{0}}, X_1 \in \mathrm{Lie}(G)_{\bar{1}}$, and $r \geq 1$, we have

$$(\mathrm{ad} X_0)^r(X_1) = \sum_{i=0}^{r} (-1)^{r-i} \binom{r}{i} X_0^i * X_1 * X_0^{r-i}. \tag{2.1}$$

This implies that $\mathrm{ad}(X_0)^p(X_1) = X_0^{[p]} * X_1 - X_1 * X_0^{[p]} = \mathrm{ad}(X_0^{[p]})(X_1)$. Thus the $\mathrm{Lie}(G)_{\bar{0}}$-module $\mathrm{Lie}(G)_{\bar{1}}$ via adjoint action is a restricted module. □

For $i = 0, 1$, we let

$$\mathrm{Der}_k(k[G], k)_{\bar{i}} := \{X \in \mathrm{Hom}_k(k[G], k)_{\bar{i}} \mid X(fg) = X(f)\varepsilon(g) + (-1)^{i|f|}\varepsilon(f)X(g)\}$$

and let $\mathrm{Der}_k(k[G], k) = \mathrm{Der}_k(k[G], k)_{\bar{0}} \oplus \mathrm{Der}_k(k[G], k)_{\bar{1}}$. The following can be established as in the case of algebraic groups (cf. [1]).

Lemma 2.4. *As restricted Lie superalgebras, Lie$(G) \cong$ Der$_k(k[G], k)$.*

2.4. **G-modules and $Dist(G)$-modules.** For any vector superspace M, we have the general linear supergroup $GL(M)$ which sends each commutative superalgebra A to $GL(M, A)$, the group of all even A-linear automorphisms of $M \otimes A$. A representation of a supergroup G or a (rational) G-module, means a natural transformation $\rho : G \to GL(M)$ for some vector superspace M. As for algebraic groups, a representation M of G is equivalent to a right $k[G]$-comodule structure on M given by an even map $\eta_M : M \to M \otimes k[G]$ (cf. [8, 3]). We denote by G-$\mathfrak{mod}$ the category of rational G-modules with (not necessarily homogeneous) G-homomorphisms. Note that a G-module is always *locally finite*, i.e., it is a sum of finite-dimensional G-modules.

Given a closed subgroup H of G, a $Dist(G)$-module M is called a $(Dist(G), H)$-module if M has also a structure of H-module such that the $Dist(H)$-module structures on M induced from the actions of $Dist(G)$ and of H coincide. We denote by $(Dist(G), H)$-$\mathfrak{mod}$ the category of locally finite $(Dist(G), H)$-modules.

There is a natural functor

$$\Psi : G\text{−}\mathfrak{mod} \longrightarrow (Dist(G), H)\text{−}\mathfrak{mod}$$

as follows: one endows a G-module M with an action of $\mu \in Dist(G)$ by $(id_M \otimes \mu) \circ \eta_M$, and M is acted by H as a subgroup of G.

2.5. **An equivalence of categories.** Let G be a supergroup. The Frobenius morphism $F : G \to G$ is the natural transformation which assigns to each $A \in \mathfrak{salg}$ the morphism $F(A) : G(A) \to G(A)$ induced by $a \mapsto a^{p^r}$ for $a \in A$. The image of F lies in $G_{\rm ev}$ and we often denote $F : G \to G_{\rm ev}$. For $r \geq 1$, we define $F^r : G \to G_{\rm ev}$ by the r-th iteration of F, whose kernel G_r is called the r-th Frobenius kernel of G. Since G_1 is a normal subgroup of G, $G_{\rm ev}G_1 := \{g_0g_1 | g_0 \in G_{\rm ev}, g_1 \in G_1\}$ is a subgroup of G.

Lemma 2.5. *For every $r \geq 1$, $G_r \cap G_{\rm ev} = (G_{\rm ev})_r$.*

Proof. Note that $G_r \cap G_{\rm ev}$ is the kernel of $F^r|_{G_{\rm ev}}$, the restriction to $G_{\rm ev}$ of the Frobenius morphism F^r on G. Now the lemma follows from that $F^r|_{G_{\rm ev}}$ coincides with the r-th Frobenius morphism of $G_{\rm ev}$. □

A key argument for the following lemma was supplied by Jon Kujawa.

Lemma 2.6. *Assume G is a supergroup with its even subgroup $G_{\rm ev}$ defined over $\mathbb{F}_p$. Then, $G = G_{\rm ev}G_1$. Furthermore, $G_r = (G_{\rm ev})_rG_1$ for every $r \geq 1$.*

Proof. The restriction $F|_{G_{\rm ev}} : G_{\rm ev} \to G_{\rm ev}$ coincides with the Frobenius morphism of $G_{\rm ev}$ which is surjective as $G_{\rm ev}$ is defined over $\mathbb{F}_p$ (cf. [8]). Recall that for any $g \in G$, we have $F(g) \in G_{\rm ev}$. By the surjectivity of the Frobenius morphism $F|_{G_{\rm ev}}$ on $G_{\rm ev}$, there exists an element $g_0 \in G_{\rm ev}$ so that $F(g_0) = F(g)$. Thus, $F(g_0^{-1}g) = 1$, i.e. $g_1 := g_0^{-1}g \in G_1$, and $g = g_0g_1 \in G_{\rm ev}G_1$.

Now if $g \in G_r$, then $1 = F^r(g) = F^r(g_0g_1) = F^r(g_0)$, that is $g_0 \in G_r$. It follows by Lemma 2.5 that $g_0 \in (G_{\rm ev})_r$. □

Since G_1 is a finite (super)group scheme and $k[G_1]$ is a finite-dimensional Hopf superalgebra, we have the next lemma following [8, Chap. I.8]. Recall for a G_1-module M, we have the comodule structure map $\eta_M : M \to M \otimes k[G_1]$. Denote by G_1-$\mathfrak{mod}$ the category of G_1-modules and by $Dist(G_1)$-$\mathfrak{mod}$ the category of $Dist(G_1)$-modules. Note that $Dist(G_1) = k[G_1]^*$.

Lemma 2.7. *The functor $\Psi_1 : G_1\text{-}\mathfrak{mod} \to Dist(G_1)\text{-}\mathfrak{mod}$, which endows a G_1-module M with an action of $\mu \in Dist(G_1)$ by $(id_M \otimes \mu) \circ \eta_M$, is an equivalence of categories.*

For the remainder of this subsection, we make the following assumption on G.

Assumptions. *G is an algebraic supergroup whose even subgroup $G_{\rm ev}$ is a connected reductive group defined over $\mathbb{F}_p$ with a maximal torus T. Furthermore, there is an integral basis for $Lie(G)$ and a corresponding basis for $Dist(G)$ which extend the Chevalley basis for $Lie(G_{\rm ev})$ and the Kostant basis for $Dist(G_{\rm ev})$ respectively.*

It is known (cf. [8]) that under the above assumption on the algebraic group $G_{\rm ev}$ the natural functor $\Psi_{\rm ev} : G_{\rm ev}\text{-}\mathfrak{mod} \to (Dist(G_{\rm ev}), T)\text{-}\mathfrak{mod}$ (defined just as the functor Ψ) is an equivalence of categories. Clearly $\Psi : G\text{-}\mathfrak{mod} \to (Dist(G), T)\text{-}\mathfrak{mod}$ is compatible with $\Psi_{\rm ev}$ and also with Ψ_1 via forgetful functors.

Theorem 2.8. *Retain the above assumptions on G. Then Ψ is an equivalence of categories between G-$\mathfrak{mod}$ and $(Dist(G), T)$-$\mathfrak{mod}$.*

Proof. We define a functor $\widetilde{\Psi}$ from $(Dist(G), T)$-$\mathfrak{mod}$ to G-$\mathfrak{mod}$ by lifting every locally finite $(Dist(G), T)$-module to a G-module as follows. Let M be a locally finite $(Dist(G), T)$-module. Noting $Dist(G_{\rm ev}) \subset Dist(G)$ as subalgebras and regarding M as a $(Dist(G_{\rm ev}), T)$-module, we can lift M to an $G_{\rm ev}$-module canonically as in [8, pp.171]. In the same vein, noting $Dist(G_1) \subset Dist(G)$ as subalgebras and regarding M as a $Dist(G_1)$-module, we can lift M to an G_1-module via an inverse functor of Ψ_1 (cf. Lemma 2.7). The M endowed with these two lifted structures coincide as $(G_{\rm ev})_1$-modules since both are lifted canonically from the same $Dist((G_{\rm ev})_1)$-module structure, and thus they coincide as $G_1 \cap G_{\rm ev}$-modules by Lemma 2.5. By Lemmas 2.5 and 2.6, we obtain a well-defined G-module structure on M by letting any $g = g_0 g_1 \in G$ with $g_0 \in G_{\rm ev}$ and $g_1 \in G_1$ acts by composing the actions of g_0 and g_1.

Clearly, Ψ and $\widetilde{\Psi}$ are inverses of each other. □

Remark 2.9. It is straightforward to check the assumptions in Theorem 2.8 for supergroups $GL(m|n)$ and $Q(n)$, cf. [3, 4]. Thus, our Theorem 2.8 gives a simpler and uniform proof of the equivalence of categories for these two supergroups obtained earlier in *loc. cit.* by first establishing an isomorphism between a restricted dual of the superalgebra of distributions and the coordinate superalgebra. We shall also see in the next section that the supergroups of type SpO satisfy these assumptions as well, and it is not clear if the method in *loc. cit.* is applicable for these new supergroups.

3. The Chevalley basis of Lie superalgebra $spo(2n|\ell)$

3.1. **The supergroup** $SpO(2n|\ell)$**.** The supergroup $\mathrm{GL}(r|s)$ is the functor which associates to any $A \in \mathfrak{salg}$ the group $GL(r|s; A)$ of all invertible $(r+s) \times (r+s)$ matrices of the form

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{3.1}$$

where a (resp. d) is an $r \times r$ (resp. $s \times s$) matrix with entries in $A_{\bar{0}}$, b, c is $r \times s$ (resp. $s \times r$) with entries in $A_{\bar{1}}$. The supertranspose of g is defined as

$$g^{st} := \begin{bmatrix} a^t & c^t \\ -b^t & d^t \end{bmatrix}$$

where the superscript t denotes the transpose of a matrix in the usual sense. It is well known (cf. e.g. [12]) that g is invertible if and only if both a and d are invertible. Let $Mat_{r|s}$ be the affine superscheme with $Mat_{r|s}(A)$ consisting of al $(r+s) \times (r+s)$ matrices of the form (3.1). Then $k[GL(r|s)]$ is the localization of $k[Mat_{r|s}]$ at the function $\det : g \to \det a \det b$. The Lie superalgebra of $\mathrm{GL}(r|s)$, denoted by $\mathfrak{gl}(r|s)$, consists of matrices of the form (3.1) with $a, b, c, d \in k$, and the $\mathbb{Z}_2$-grading is defined such that $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ is even and $\begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}$ is odd.

Recall (cf. [12, Chap. 3]) there is a morphism of supergroups called the *Berezian or superdeterminant*, $\mathrm{Ber} : GL(r|s) \to GL(1|0)$ defined as follows: for any $A \in \mathfrak{salg}$,

Ber : $GL(r|s;A) \to GL(1|0;A)$ sends an element g in (3.1) to

$$\mathrm{Ber}(g) = \det(a - bd^{-1}c) \cdot \det d^{-1}.$$

It has various favorable properties, e.g., $\mathrm{Ber}(g^{st}) = \mathrm{Ber}(g)$.

Define the $(2n+2m+1)\times(2n+2m+1)$ matrix in the $(n|n|m|m|1)$-block form

$$\mathfrak{J}_{2n|2m+1} := \begin{bmatrix} 0 & I_n & 0 & 0 & 0 \\ -I_n & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_m & 0 \\ 0 & 0 & I_m & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.2}$$

where I_n is the $n\times n$ identity matrix. Let $\mathfrak{J}_{2n|2m}$ denote the $(2n+2m)\times(2n+2m)$ matrix obtained from $\mathfrak{J}_{2n|2m+1}$ with the last row and column deleted. Denote by $SpO(2n|\ell)$ (with $\ell = 2m$ or $2m+1$) the supergroup functor which associates to any $A \in \mathfrak{salg}$ the group which consists of all $(2n+\ell)\times(2n+\ell)$ matrices of the form

$$\{g \in GL(2n|\ell;A) \mid g^{st}\mathfrak{J}_{2n|\ell}\, g = \mathfrak{J}_{2n|\ell}, \mathrm{Ber}(g) = 1\}. \tag{3.3}$$

Note that the defining relations in (3.3) are actually defined over $\mathbb{F}_p$. The underlying even subgroup is

$$SpO(2n|\ell)_{\mathrm{ev}} = SpO(2n|\ell) \cap GL(2n|\ell)_{\mathrm{ev}} \cong Sp(2l)\times SO(\ell).$$

3.2. **Lie superalgebra** $spo(2n|\ell)$**.** As in the case of Lie algebras, with the help of Lemma 2.4 we can identify the Lie algebra $\mathrm{Lie}(SpO(2n|\ell))$ with

$$spo(2n|\ell) := \{g \in \mathfrak{gl}(2n|\ell) \mid g^{st}\mathfrak{J}_{2n|\ell} + \mathfrak{J}_{2n|\ell}\, g = 0\}.$$

The $spo(2n|2m+1)$ consists of the $(2n+2m+1)\times(2n+2m+1)$ matrices in the following $(n|n|m|m|1)$-block form

$$g = \begin{bmatrix} d & e & y_1^t & x_1^t & z_1^t \\ f & -d^t & -y^t & -x^t & -z^t \\ x & x_1 & a & b & -v^t \\ y & y_1 & c & -a^t & -u^t \\ z & z_1 & u & v & 0 \end{bmatrix} \tag{3.4}$$

where b, c are skew-symmetric, and e, f are symmetric matrices. The Lie superalgebra $spo(2n|2m+1)$ is called type $B(m,n)$ in [9]. We assume here and below to index the rows and columns of (3.2) and (3.4) by the finite set $I(2n|2m+1)$, where

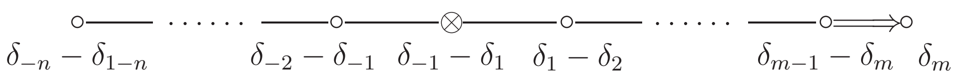
$$I(r|s) := \{-r, -r+1, \cdots, -1; 1, 2, \cdots, s\}.$$

Similarly, the Lie superalgebra $spo(2n|2m)$ consists of the $(2n+2m)\times(2n+2m)$ matrices which are obtained from g of the form (3.4) with the last row/column deleted and whose rows/columns are indexed by $I(2n|2m)$.

3.3. **Chevalley basis for $spo(2n|2m+1)$.** The even subalgebra of $spo(2n|2m+1)$ is $sp(2n) \oplus so(2m+1)$. The Cartan subalgebra $\mathfrak{h}$ of $spo(2n|2m+1)$ is taken to be the subalgebra of diagonal matrices, and it has a basis given by $E_{i-n,i-n} - E_{i,i}, E_{j,j} - E_{j+m,j+m}(-n \le i < 0 < j \le m)$, and let δ_a, where $a \in I(n|m)$, be the dual basis. The standard set Π of simple roots is:

$$\{\delta_{-n} - \delta_{-n+1}, \dots, \delta_{-2} - \delta_{-1}, \delta_{-1} - \delta_1, \delta_1 - \delta_2, \dots, \delta_{m-1} - \delta_m, \delta_m\} \text{ for } m \ge 1,$$

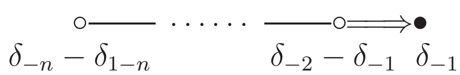
with $\delta_{-1} - \delta_1$ being odd. The corresponding Dynkin diagram is (where the node $\otimes$ denotes an odd simple root twice of which is not root):



If $m = 0$, then the standard set Π of simple roots is

$$\Pi = \{\delta_{-n} - \delta_{-n+1}, \dots, \delta_{-2} - \delta_{-1}, \delta_{-1}\},$$

and its Dynkin diagram is (where the node $\bullet$ denotes an odd simple root twice of which is a root):



The set of roots are $\Delta = \Delta_0 \cup \Delta_1$, a union of sets of even roots Δ_0 and odd roots Δ_1, and $\Delta = \Delta^+ \cup -\Delta^+$. Set $\Delta_i = \Delta_i^+ \cup -\Delta_i^+$ for $i = 0, 1$. More explicitly,

$$\begin{aligned} \Delta_0^+ &= \{\delta_i \pm \delta_j, -n \le i < j < 0 \text{ or } 0 < i < j \le m\} \\ &\quad \cup \{2\delta_i, -n \le i < 0\} \cup \{\delta_j, 0 < j \le m\}, \\ \Delta_1^+ &= \{\delta_i \pm \delta_j, -n \le i < 0 < j \le m\} \cup \{\delta_i, -n \le i < 0\}. \end{aligned}$$

Note that Δ^+ is compatible with a set of positive roots of $sp(2n) \oplus so(2m+1)$ whose fundamental system is given by:

$$\Pi_{\text{ev}} := \{\delta_i - \delta_{i+1}, -n \le i < -1; 2\delta_{-1}\} \cup \{\delta_i - \delta_{i+1}, 1 \le i < m; \delta_m\}.$$

It is understood that for $m = 0$ the undefined δ_0 is omitted from Π_{ev}.

Recall (cf. [17]) that $sp(2n) \oplus so(2m+1)$ admits a Chevalley basis (unique up to signs) $\{H_s(s \in \Pi_{\text{ev}}), X_\alpha(\alpha \in \Delta_0)\}$, whose structure constants are integers. One of the requirements [17, Theorem 1] is that

$$[X_\alpha, X_\beta] = \pm(r+1)X_{\alpha+\beta}, \text{ if } \alpha, \beta, \alpha+\beta \in \Delta_0, \tag{3.5}$$

where r is determined by the α-string of roots through β: $\{\beta + i\alpha \mid -r \le i \le q\}$.

Remark 3.1. Associated to the even short root δ_j with $0 < j \leq m$, we fix the sign and take the Chevalley root vector $X_{\delta_j} = \sqrt{2}(E_{j,2m+1} - E_{2m+1,j+m})$ where $\sqrt{2} \in k$. The other Chevalley (long) root vectors for $sp(2n) \oplus so(2m+1)$ always have entries $0, \pm 1$ in the standard matrix form (3.4).

Now we define the odd root vectors (indexed by their corresponding roots): for $-n \leq i < 0 < j \leq m$,

$$X_{\delta_i+\delta_j} := E_{j,i} + E_{i-n,j+m}, \qquad X_{-\delta_i-\delta_j} := E_{j+m,i-n} - E_{i,j}, \tag{3.6}$$

$$X_{\delta_i-\delta_j} := E_{j+m,i} + E_{i-n,j}, \qquad X_{-\delta_i+\delta_j} := E_{j,i-n} - E_{i,j+m}, \tag{3.7}$$

$$X_{\delta_i} := \sqrt{2}(E_{2m+1,i} + E_{i-n,2m+1}), \qquad X_{-\delta_i} := \sqrt{2}(E_{2m+1,i-n} - E_{i,2m+1}). \tag{3.8}$$

Proposition 3.2. *The Chevalley basis elements $\{H_s\,(s \in \Pi_{\rm ev}), X_\alpha\,(\alpha \in \Delta_0)\}$ for the even subalgebra $sp(2n) \oplus so(2m+1)$ together with $X_\alpha\,(\alpha \in \Delta_1)$ in (3.6-3.8) form a basis for Lie superalgebra $spo(2n|2m+1)$ with integer structure constants.*

This basis will be called the *Chevalley basis* for $spo(2n|2m+1)$.

Proof. Follows by a direct computation. □

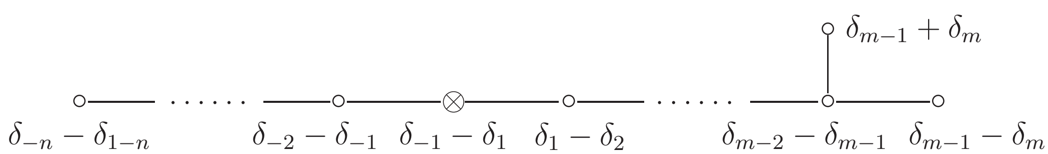
3.4. **Chevalley basis for** $spo(2n|2m), m \geq 2$**.** We continue to regard Lie superalgebra $spo(2n|2m)$ as a subalgebra of Lie superalgebra $spo(2n|2m+1)$ in the matrix form (3.4). The even subalgebra of $spo(2n|2m)$ is $sp(2n) \oplus so(2m)$. The Cartan subalgebra of $spo(2n|2m)$ is the same as the Cartan subalgebra of $spo(2n|2m+1)$. The standard set $\Pi_{\rm ev}$ of simple roots for $sp(2n) \oplus so(2m)$ is

$$\Pi_{\rm ev} := \{\delta_i - \delta_{i+1}, -n \leq i < -1; 2\delta_{-1}\} \cup \{\delta_i - \delta_{i+1}, 1 \leq i < m; \delta_{m-1} + \delta_m\}.$$

Lie superalgebra $spo(2n|2m)$ with $m \geq 2$ (and $n \geq 1$) is called type $D(m,n)$ in [9]. Its standard set Π of simple roots is:

$$\{\delta_{-n} - \delta_{1-n}, \cdots, \delta_{-2} - \delta_{-1}, \delta_{-1} - \delta_1, \delta_1 - \delta_2, \cdots, \delta_{m-1} - \delta_m, \delta_{m-1} + \delta_m\},$$

with $\delta_{-1} - \delta_1$ being odd. Its Dynkin diagram is



The corresponding set of positive roots $\Delta^+ = \Delta_0^+ \cup \Delta_1^+$, where

$$\Delta_0^+ = \{\delta_i \pm \delta_j, -n \leq i < j < 0 \text{ or } 0 < i < j \leq m\} \cup \{2\delta_i, -n \leq i < 0\},$$

$$\Delta_1^+ = \{\delta_i \pm \delta_j, -n \leq i < 0 < j \leq m\}.$$

The set of roots is $\Delta = \Delta^+ \cup -\Delta^+$. Set $\Delta_i = \Delta_i^+ \cup -\Delta_i^+$ for $i = 0, 1$.

Proposition 3.3. *Let $m \geq 2$. The Chevalley basis elements for the even subalgebra $sp(2n) \oplus so(2m)$ together with the odd root vectors $X_\alpha\,(\alpha \in \Delta_1)$ in (3.6-3.7) form a basis for Lie superalgebra $spo(2n|2m)$ with integer structure constants.*

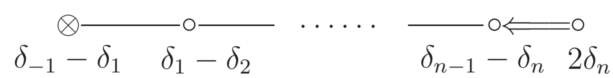
The proof of this proposition is again by a direct computation. This basis will be called the *Chevalley basis* for $spo(2n|2m)$.

3.5. **Chevalley basis for** $spo(2n|2)$**.** Lie superalgebra $spo(2n|2)$ is called type $C(n)$ in [9], and it is more convenient to be realized as $osp(2|2n)$ which consists of matrices of the $(1|1|n|n)$-block form

$$\begin{bmatrix} a & 0 & x & x_1 \\ 0 & -a & y & y_1 \\ y_1^t & x_1^t & d & e \\ -y^t & -x^t & f & -d^t \end{bmatrix}$$

where e, f are symmetric matrices. We index the rows and columns by $I(2|2n)$. The even subalgebra is $so(2) \oplus sp(2n)$. The Cartan subalgebra $\mathfrak{h}$ has a basis given by $E_{-2,-2} - E_{-1,-1}, E_{i,i} - E_{n+i,n+i} (1 \le i \le n)$, and let $\delta_i \in \mathfrak{h}^*(i \in I(1|n))$ be the corresponding dual basis.

A standard set Π of simple roots is $\{\delta_{-1} - \delta_1, \delta_1 - \delta_2, \cdots, \delta_{n-1} - \delta_n, 2\delta_n\}$ with $\delta_{-1} - \delta_1$ being odd. The Dynkin diagram with respect to Π is



The set of positive roots is $\Delta^+ = \Delta_0^+ \cup \Delta_1^+$, where

$$\Delta_0^+ = \{\delta_i \pm \delta_j, 1 \le i < j \le n\} \cup \{2\delta_i, 1 \le i \le n\},$$
$$\Delta_1^+ = \{\delta_{-1} \pm \delta_i, 1 \le i \le n\}.$$

We define the odd root vectors: for $1 \le j \le n$,

$$X_{\delta_{-1}+\delta_j} := E_{-2,j+n} + E_{j,-1}, \qquad X_{-\delta_{-1}-\delta_j} := E_{j+n,-2} - E_{-1,j}, \tag{3.9}$$
$$X_{\delta_{-1}-\delta_j} := E_{-2,j} - E_{j+n,-1}, \qquad X_{-\delta_{-1}+\delta_j} := E_{-1,j+n} + E_{j,-2}. \tag{3.10}$$

Proposition 3.4. *The Chevalley basis elements for $sp(2n)$, the vector $E_{-2,-2} - E_{-1,-1}$ in $so(2)$, together with the odd root vectors X_α $(\alpha \in \Delta_1)$ in (3.9-3.10) form a basis for Lie superalgebra $spo(2n|2)$ with integer structure constants.*

The proof of this proposition is again by a direct computation. This basis will be called the *Chevalley basis* for $spo(2n|2)$.

Remark 3.5. The Chevalley basis for $spo(2n|\ell)$ introduced in this paper has the following remarkable property (which can be verified case by case):

$$[X_\alpha, X_\beta] = \pm(r+1)X_{\alpha+\beta}, \text{ if } \alpha, \beta, \alpha+\beta \in \Delta,$$

where r is determined by the α-string of roots through β: $\{\beta + i\alpha \mid -r \le i \le q\} \subseteq \Delta \cup \{0\}$. Of course, this is part of the definition of the Chevalley basis for the even subalgebra of $spo(2n|\ell)$, and it makes no difference in this case (and most other cases) to have $\Delta \cup \{0\}$ here instead of the usual Δ. The additional $\{0\}$ is needed

exactly when $\ell = 2m+1$ and $\alpha = \beta = \pm\delta_i$ for $-n \le i < 0$, where twice a root happen to be a root.

This observation suggests a possible uniform definition of Chevalley basis for all simple Lie superalgebras of the classical types (which has been classified [9]).

3.6. **Basis of the superalgebra of distributions.** Let $G = SpO(2n|\ell)$ and $\mathfrak{g} = spo(2n|\ell)$ with Chevalley basis $\{X_\alpha, H_s, | \alpha \in \Delta, s \in \Pi_{\rm ev}\}$. Denote by $\mathfrak{g}_{\mathbb{C}}$ the complex Lie superalgebra $spo(2n|\ell, \mathbb{C})$. The PBW theorem for Lie superalgebra implies that the universal enveloping superalgebra $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ has a basis

$$\prod_{\alpha\in\Delta_0} X_\alpha^{n_\alpha} \prod_{s\in\Pi_{\rm ev}} H_s^{m_s} \prod_{\beta\in\Delta_1} X_\beta^{\varepsilon_\beta}$$

with $n_\alpha, m_s \in \mathbb{Z}_+$ and $\varepsilon_\beta \in \{0,1\}$. Define the Kostant $\mathbb{Z}$-form $\mathcal{U}_{\mathbb{Z}}$ to be the $\mathbb{Z}$-subalgebra of $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ generated by the following elements

$$X_\alpha^{(r)} := \frac{X_\alpha^r}{r!}, \ \ \alpha \in \Delta_0, r \in \mathbb{Z}_+; \quad X_\beta, \ \ \beta \in \Delta_1;$$
$$\binom{H_s}{m_s} := \prod_{j=0}^{m_s-1} (H_s - j)/m_s!, \ \ s \in \Pi_{\rm ev}, m_s \in \mathbb{Z}_+.$$

We define $\mathcal{U}_k := \mathcal{U}_{\mathbb{Z}} \otimes_{\mathbb{Z}} k$.

Theorem 3.6. (a) *The superalgebra $\mathcal{U}_{\mathbb{Z}}$ is a free $\mathbb{Z}$-module with basis*

$$\prod_{\alpha\in\Delta_0} X_\alpha^{(n_\alpha)} \prod_{s\in\Pi_{\rm ev}} \binom{H_s}{m_s} \prod_{\beta\in\Delta_1} X_\beta^{\varepsilon_\beta} \tag{3.11}$$

for $n_\alpha, m_s \in \mathbb{Z}_+$ and $\varepsilon_\beta \in \{0,1\}$, where the product is taken in any fixed order.

(b) *As Hopf superalgebras, $\mathcal{U}_k$ is isomorphic to Dist(G).*

Proof. The isomorphism in (b) is the reduction modulo p of the isomorphism over $\mathbb{C}$ just as for algebraic groups, cf. [8, II.1.12]; also see [3, 4] for the supergroups of type Q and A.

The proof of (a) basically follows the proof of [17, Theorem 2], with the following additional computation regarding the odd root vectors. First, we have for $\beta \in \Delta_1$ that

$$X_\beta \binom{H_s}{t} = \sum_{r=0}^{t} (-1)^{t-r} \binom{\beta(H_s)}{t-r} \binom{H_s}{r} X_\beta$$

where the coefficients on the right-hand side are integers thanks to a Chevalley basis property $\beta(H_s) \in \mathbb{Z}$.

Given $\alpha \in \Delta_0, \beta \in \Delta_1$, we have $(\mathrm{ad} X_\alpha)^3 X_\beta = 0$ since $\beta + 3\alpha$ is never a root for $spo(2n|\ell)$ by inspection. If $(\mathrm{ad} X_\alpha)^2 X_\beta = 0$, then

$$X_\beta X_\alpha^{(r)} = X_\alpha^{(r)} X_\beta + \sum_{\nu\in\Delta_1} c_{\beta,\alpha}^{\nu} X_\alpha^{(r-1)} X_\nu$$

where $[X_\beta, X_\alpha] = \sum_{\nu\in\Delta_1} c^\nu_{\beta,\alpha} X_\nu$ with all $c^\nu_{\beta,\alpha} \in \mathbb{Z}$ by the construction of Chevalley basis for $spo(2n|\ell)$.

By inspection, the inequality $(\mathrm{ad} X_\alpha)^2 X_\beta \neq 0$ occurs exactly when $\ell = 2m+1$, $\alpha = \delta_j$ and $\beta = \pm\delta_i - \delta_j$ (or the pair of α, β with a simultaneous change of signs), where $-n \leq i < 0 < j \leq m$. By Remark 3.1 and the explicit formulas for odd root vectors, we have

$$\begin{aligned} X_\beta X_\alpha^{(r)} &= X_\alpha^{(r)} X_\beta + X_\alpha^{(r-1)} [X_\beta, X_\alpha] + X_\alpha^{(r-2)} \cdot \frac{1}{2} (\mathrm{ad} X_\alpha)^2 X_\beta \\ &= X_\alpha^{(r)} X_\beta + X_\alpha^{(r-1)} X_{\pm\delta_i} - X_\alpha^{(r-2)} X_{\pm\delta_i+\delta_j}. \end{aligned}$$

We observe that all the coefficients on the right-hand side are integers. □

From now on, we will always identify $\mathrm{Dist}(G)$ with $\mathcal{U}_k$ (and also the corresponding subalgebras between them).

3.7. **Various subalgebras.** Let T be the maximal torus of $G = SpO(2n|\ell)$ which consists of the diagonal matrices and B be the Borel subgroup corresponding to Δ^+. Then $Dist(T)$ has a basis given by $\binom{H_s}{m_s}$ for $m_s \in \mathbb{Z}_+$, $s \in \Pi_{\mathrm{ev}}$, and $Dist(B)$ has a basis given by

$$\prod_{\alpha\in\Delta_0^+} X_\alpha^{(n_\alpha)} \prod_{s\in\Pi_{\mathrm{ev}}} \binom{H_s}{m_s} \prod_{\beta\in\Delta_1^+} X_\beta^{\varepsilon_\beta} \tag{3.12}$$

for $n_\alpha, m_s \in \mathbb{Z}_+$ and $\varepsilon_\beta \in \{0,1\}$.

Clearly, the Frobenius morphism F^r preserves the various subgroups T, B_{ev}, B of G, for $r \geq 1$. We denote by $\mathcal{U}_r$ the k-subalgebra of $\mathcal{U}_k$ spanned by the elements (3.11) with $0 \leq n_\alpha < p^r$, $0 \leq m_s < p^r$ and $\varepsilon_\beta \in \{0,1\}$. As in the purely even case (cf. [8]), $\mathrm{Dist}(G_r) \cong \mathcal{U}_r$, and we will not distinguish these two superalgebras. In particular, $\mathrm{Dist}(G_1)$ is the restricted enveloping superalgebra of $\mathfrak{g}$. Similarly, a basis for $Dist(B_r)$ of the r-th Frobenius kernel B_r is given by the elements (3.12) for $0 \leq n_\alpha < p^r$, $0 \leq m_s < p^r$ and $\varepsilon_\beta \in \{0,1\}$.

4. A tensor product theorem for $SpO(2n|\ell)$

Let $G = SpO(2n|\ell)$. The equivalence of categories established in Theorem 2.8 allows us to study G-modules via a highest weight theory of $Dist(G)$-modules. The proofs in this section are similar to [3, 4, 10] for $Q(n)$ and $GL(m|n)$, which in turn were super modification from standard developments in the algebraic group setup (cf. [5, 8]).

4.1. **Highest weight modules.** We continue to denote $G = SpO(2n|\ell)$ with maximal torus T and Borel subgroup B as before. The character group is

$$X(T) = \left\{ \Sigma_{i\in I(n|m)} \lambda_i \delta_i \mid \text{ all } \lambda_i \in \mathbb{Z} \right\}.$$

A standard symmetric bilinear form on $X(T)$ is defined by

$$(\delta_i, \delta_j) = \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j < 0 \\ -1, & \text{if } i = j > 0. \end{cases}$$

Denote $Y(T) = \mathrm{Hom}(\mathbb{G}_m, T)$. There is a natural pairing $\langle\ ,\ \rangle : X(T) \times Y(T) \to \mathbb{Z}$. Given an even root $\alpha \in \Delta_0$, its coroot $\alpha^\vee \in Y(T)$ is defined as in [8]. The Weyl group W is generated by the reflections s_α for $\alpha \in \Delta_0$, where $s_\alpha \lambda = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$ for $\lambda \in X(T)$. The set

$$X^+(T) := \left\{ \lambda = \Sigma_{i \in I(n|m)} \lambda_i \delta_i \in X(T) \mid \langle \lambda, \alpha^\vee \rangle \geq 0 \text{ for all } \alpha \in \Delta_0^+ \right\}$$

can be easily identified with

$$\{\lambda \in X(T) \mid \lambda_{-n} \geq \cdots \geq \lambda_{-1} \geq 0, \lambda_1 \geq \cdots \geq \lambda_m \geq 0\}, \text{ if } \ell = 2m+1,$$

and identified with

$$\{\lambda \in X(T) \mid \lambda_{-n} \geq \cdots \geq \lambda_{-1} \geq 0, \lambda_1 \geq \cdots \geq \lambda_{m-1} \geq |\lambda_m| \geq 0\}, \text{ if } \ell = 2m.$$

For an $\lambda \in X(T)$ and a $\mathrm{Dist}(G)$-module M, the λ-weight subspace of M is

$$M_\lambda = \left\{ m \in M \mid \binom{H_s}{m_s} m = \binom{\langle \lambda, H_s \rangle}{m_s} m, \ \forall s \in \Pi_{\mathrm{ev}}, m_s \in \mathbb{Z}_+ \right\}.$$

For $\lambda \in X(T)$, we denote the Verma module

$$M(\lambda) = Dist(G) \otimes_{Dist(B)} k_\lambda$$

where k_λ is the one-dimensional $Dist(B)$-module of weight λ (in degree $\bar{0}$). It is standard to see that the $Dist(G)$-module $M(\lambda)$ has a unique simple quotient $L(\lambda)$ and that the $Dist(G)$-modules $L(\lambda)$, $\lambda \in X(T)$, are pairwise non-isomorphic. By definition, $L(\lambda)$ is $X(T)$-graded and thus a T-module.

Lemma 4.1. *Every simple module M in the category $(Dist(G), T)$-$\mathfrak{mod}$ is isomorphic to a finite-dimensional highest weight module $L(\lambda)$ for some $\lambda \in X^+(T)$.*

Proof. Since M is simple and locally finite, it is finite-dimensional. By weight consideration, there exists a highest weight vector v_λ of weight $\lambda \in X(T)$ such that $X_\alpha^{(r)} v_\lambda = 0$ for all $\alpha \in \Delta^+, r \geq 1$, (and $r = 1$ for α odd). It follows that $M = L(\lambda)$. Now $Dist(G_{\mathrm{ev}}).v_\lambda$ as a subspace of $L(\lambda)$ is finite dimensional. A classical result applied to G_{ev} asserts that $\lambda \in X^+(T)$. □

A major challenge here, which is a super phenomenon and does not occur for $GL(m|n)$, is that $L(\lambda)$ for various $\lambda \in X^+(T)$ may fail to be finite-dimensional in general. By the equivalence of categories G-$\mathfrak{mod} \cong (Dist(G), T)$-$\mathfrak{mod}$ in Theorem 2.8, we will no longer distinguish a G-module M from a $(Dist(G), T)$-module M. Define

$$X^\dagger(T) = \{\lambda \in X^+(T) \mid L(\lambda) \text{ is finite-dimensional}\}. \tag{4.1}$$

By Lemma 4.1, the $L(\lambda)$'s, where λ runs over $X^\dagger(T)$, form a complete list of pairwise non-isomorphic simple G-modules. Thus, the classification of simple G-modules boils down to the nontrivial problem of determining explicitly the set $X^\dagger(T)$, which will be solved completely in Section 5.

4.2. **The G_r-modules.** Given $\lambda \in X(T)$, define the baby Verma module (which is also a G_r-module)

$$Z_r(\lambda) := Dist(G_r) \otimes_{Dist(B_r)} k_\lambda.$$

One can show as usual that $Z_r(\lambda)$ has a unique simple G_r-quotient, which will be denoted by $L_r(\lambda)$. The following is standard.

Proposition 4.2. *Every simple G_r-module is isomorphic to $L_r(\lambda)$ for some $\lambda \in X(T)$. Furthermore, $L_r(\lambda) \cong L_r(\mu)$ if and only if $\lambda - \mu \in p^r X(T)$.*

4.3. **A tensor product theorem.**

Proposition 4.3. *Let $r \geq 1$.*

1. *Every simple G-module regarded as a G_r-module is completely reducible.*
2. *For $\lambda \in X^\dagger(T)$ and $0 \neq v_\lambda \in L(\lambda)_\lambda$, $Dist(G_r).v_\lambda$ is a G_r-submodule of $L(\lambda)$ and it is isomorphic to $L_r(\lambda)$.*

Proof. Let M be a simple G-module. Take any simple G_r-submodule M_1 in M. Then, $\sum_{g \in G_{\mathrm{ev}}(k)} gM_1$ is a completely reducible G_r-module; it is also a G_{ev}-module and a G_1-module, hence a G-module by Lemma 2.6. Thus, $M = \sum_{g \in G_{\mathrm{ev}}(k)} gM_1$.

From the B_r-homomorphism $k_\lambda \to L(\lambda)$ and Frobenius reciprocity, we obtain a G_r-homomorphism $Z_r(\lambda) \to L(\lambda)$ with image $Dist(G_r).v_\lambda$. Now (2) follows from that $Dist(G_r).v_\lambda$ is completely reducible by (1) and that $Z_r(\lambda)$ has a simple G_r-quotient $L_r(\lambda)$. □

Denote $X_r(T) = \{\lambda \in X(T) \mid 0 \leq \langle \lambda, \alpha^\vee \rangle < p^r \text{ for all } \alpha \in \Pi_{\mathrm{ev}}\}$.

Proposition 4.4. *For $\lambda \in X_r(T) \cap X^\dagger(T)$, the restriction to G_r of the simple G-module $L(\lambda)$ remains to be simple, and is isomorphic to $L_r(\lambda)$.*

Proof. By Proposition 4.3, the G_r-module $M := Dist(G_r).v_\lambda$ is isomorphic to $L_r(\lambda)$. It remains to show that M is a G-module, or to show that M is a $Dist(G)$-module. Since $Dist(G)$ is generated by $Dist(G_{\mathrm{ev}})$ and $Dist(G_1)$, it suffices to show that M is preserved by the action of $Dist(G_{\mathrm{ev}})$. Since M is a G_r-module and B_{ev} normalizes G_r, M is already preserved by the action of B_{ev} or $Dist(B_{\mathrm{ev}})$. So it remains to show that M is preserved by the action of $X_\alpha^{(m)}$ for every $m \geq 1$ and negative even roots α. This can be proved by the inductive argument used for the supergroup $GL(m|n)$ in [10, Lemma 6.3] (which goes back to Borel according to [3, Lemma 9.8] for $Q(n)$). We omit the details. □

Let us denote by $L_{\mathrm{ev}}(\lambda)$ the simple G_{ev}-module of highest weight $\lambda \in X^+(T)$. The pullback $L_{\mathrm{ev}}(\lambda)^{[r]} := (F^r)^* L_{\mathrm{ev}}(\lambda)$ as a G-module is clearly simple and isomorphic to $L(p^r\lambda)$. We have the following analogue of the Steinberg tensor product theorem [16], which will become more precise with the determination of the set $X^\dagger(T)$ in the next section.

Theorem 4.5. (1) *For $r \geq 1$, $\lambda \in X_r(T) \cap X^\dagger(T)$, and $\mu \in X^+(T)$, we have*
$$L(\lambda + p^r\mu) \cong L(\lambda) \otimes L_{\mathrm{ev}}(\mu)^{[r]}.$$
(2) *Let $\lambda = \sum_{i=0}^m p^i \lambda^{(i)}$ be such that $\lambda^{(0)} \in X_1(T) \cap X^\dagger(T)$ and $\lambda^{(i)} \in X_1(T)$ for $1 \leq i \leq m$. Then*
$$L(\lambda) \cong L(\lambda^{(0)}) \otimes L_{\mathrm{ev}}(\lambda^{(1)})^{[1]} \otimes \cdots \otimes L_{\mathrm{ev}}(\lambda^{(m)})^{[m]}.$$

Proof. Both statements follow easily with the help of the classical Steinberg tensor product theorem for G_{ev}, once we establish the special case of (1) with $r = 1$. This case can be proved using Propositions 4.3 and 4.4 in the same way as for algebraic groups [8] (cf. [3, 10]). □

5. Classification of simple SpO-modules

5.1. **The cases of $SpO(2n|1)$ and $SpO(2n|2)$.** Among all $spo(2n|\ell)$ with $n > 0$ and $\ell > 0$, $spo(2n|1)$ and $spo(2n|2)$ distinguish themselves from the others in that they are three-component $\mathbb{Z}$-graded Lie superalgebra: $\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_{+1}$, where $\mathfrak{g}_0$ coincides with the even subalgebra $\mathfrak{g}_{\bar{0}}$, and $\mathfrak{g}_{\pm 1}$ is generated by root vectors associated to positive/negative odd roots.

Proposition 5.1. *Let $n > 0$ and $\ell = 1$ or 2. Then the $L(\lambda)$, where λ runs over $X^+(T)$, form a complete list of pairwise non-isomorphic simple $SpO(2n|\ell)$-modules.*

Proof. The proof is the same as in the case of characteristic zero (cf. [9]). Let $G = SpO(2n|\ell)$. By Lemma 4.1, every simple G-module is of the form $L(\lambda)$ for some $\lambda \in X^+(T)$. Because of the three-component $\mathbb{Z}$-grading on $\mathfrak{g} = spo(2n|\ell)$, one has a decomposition $Dist(G) = Dist(G)_{-1} \cdot Dist(G_{\mathrm{ev}}) \cdot Dist(G)_{+1}$, where $Dist(G)_{\pm 1}$ is generated by the odd positive/negative root vectors and $Dist(G_{\mathrm{ev}}) \cdot Dist(G)_{+1}$ is a subalgebra. Then for every $\lambda \in X^+(T)$, the irreducible $Dist(G_{\mathrm{ev}})$-module $L_{\mathrm{ev}}(\lambda)$ extends trivially to a $Dist(G_{\mathrm{ev}}) \cdot Dist(G)_{+1}$-module. Hence, $K(\lambda) := Dist(G) \otimes_{Dist(G_{\mathrm{ev}}) \cdot Dist(G)_{+1}} L_{\mathrm{ev}}(\lambda)$ is finite-dimensional of highest weight λ, with $L(\lambda)$ as its irreducible quotient. □

5.2. **Combinatorics related to Mullineux conjecture.** Let $\mu = (\mu_1, \mu_2, \ldots)$ be a partition. We will identify it with its Young diagram and denote by $\ell(\mu)$ its length. The *rim* of the Young diagram μ is the set of cells (i, j) such that the cell $(i+1, j+1)$ is not in μ. The *p-rim* is a subset of the rim defined as follows in terms of p-segments. The first p-segment consists of the first p cells in the rim from the left. The second p-segment starts with the first cell in the rim strictly to the right of the previous segment, and so on. The last p-segment is allowed to consist of possibly cells fewer than p.

A cell of μ is *p-removable* if it is at the end of a row of μ and it is in the p-rim but not at the end of any p-segment. Denote by $J(\mu)$ the partition obtained from μ by deleting all p-removable cells of μ. The number, denoted by $j(\mu)$, of all p-removable cells in μ is then given by
$$j(\mu) = |\mu| - |J(\mu)|.$$

Let us denote $J^i(\mu) = J(J^{i-1}(\mu))$ for $i \geq 1$, with $J^0(\mu) = \mu$. A partition $\mu = (\mu_1, \mu_2, \ldots)$ is *p-restricted* (or simply *restricted*) if either $p = 0$ or $p > 0$ and $\mu_i - \mu_{i+1} < p$ for all $i \geq 1$. Note that the notions j and J make sense for arbitrary partitions, and for any partitions μ and ν,

$$j(\mu + p\nu) = j(\mu). \tag{5.1}$$

Let $\mathcal{RP}(d)$ be the set of restricted partition of d, and let $\mathcal{RP} = \bigsqcup_{d \geq 0} \mathcal{RP}(d)$.

For example, let $\mu = (5, 4, 3, 3, 1, 1)$ and $p = 5$. The p-rim of μ consists of two p-segments and it is consisted of the cells colored in black (including both $\bullet$ and the double-circled cells) as follows, where the p-removable cells are the double-circled ones. Hence $j(\mu) = 4$, and by deleting the double-circled cells from μ we obtain that $J(\mu) = (4, 3, 3, 3)$.

$$\begin{array}{lllll}
\circ & \circ & \circ & \bullet & \odot \\
\circ & \circ & \circ & \odot & \\
\circ & \circ & \circ & & \\
\bullet & \bullet & \bullet & & \\
\odot & & & & \\
\odot & & & &
\end{array}$$

Theorem 5.2. *For $\mu \in \mathcal{RP}$, set $\lambda_i = j(J^{i-1}(\mu))$ for $i \geq 1$. Then, sending $\mu \in \mathcal{RP}$ to $\mathtt{M}(\mu) := (\lambda_1, \lambda_2, \cdots)$ defines a bijection $\mathtt{M} \colon \mathcal{RP} \to \mathcal{RP}$ which satisfies $\mathtt{M} = \mathtt{M}^{-1}$.*

It is well known that the simple kS_d-modules D_μ are parameterized by $\mu \in \mathcal{RP}(d)$, which are the transposes of the p-regular partitions. The Mullineux conjecture [13] (now a theorem due to [6]) states that $D_\mu \otimes \mathrm{sgn} \cong D_{\mathtt{M}(\mu)}$, where sgn is the one-dimensional sign kS_n-module. The formulation in Theorem 5.2 due to [18] of the *Mullineux bijection* $\mathtt{M} : \mathcal{RP}(d) \to \mathcal{RP}(d)$ is equivalent to the earlier formulations by Mullineux, Kleshchev and others (cf. [4] for references and history).

5.3. **The classification.** Recall T is the maximal torus of $SpO(2n|\ell)$, with $\ell = 2m$ or $2m + 1$. A weight $\lambda \in X(T)$ can be expressed as $\lambda = \sum_{i \in I(n|m)} \lambda_i \delta_i$.

Recall that the simple G-modules have been classified for $G = SpO(2n|1)$ and $SpO(2n|2)$. (The case of $SpO(2n|1)$ also fits into the general statement below.)

Theorem 5.3. (1) *Let $n \geq 1$. A complete list of pairwise non-isomorphic simple $SpO(2n|2m+1)$-modules is $\{L(\lambda)\}$, where λ runs over the set:*

$$\begin{aligned}\big\{ \Sigma_{i \in I(n|m)} \lambda_i \delta_i \mid \lambda_{-n} \geq \cdots \geq \lambda_{-1} \geq 0, \lambda_1 \geq \cdots \geq \lambda_m \geq 0, \\ \textit{all } \lambda_i \in \mathbb{Z},\ j(\lambda_1, \ldots, \lambda_m) \leq \lambda_{-1} \big\}.\end{aligned}$$

(2) *Let $m \geq 2$ and $n \geq 1$. A complete list of pairwise non-isomorphic simple $SpO(2n|2m)$-modules is $\{L(\lambda)\}$, where λ runs over the following set:*

$$\begin{aligned}\big\{ \Sigma_{i \in I(n|m)} \lambda_i \delta_i \mid \lambda_{-n} \geq \cdots \geq \lambda_{-1} \geq 0, \lambda_1 \geq \cdots \geq \lambda_{m-1} \geq |\lambda_m| \geq 0, \\ \textit{all } \lambda_i \in \mathbb{Z},\ j(\lambda_1, \ldots, |\lambda_m|) \leq \lambda_{-1} \big\}.\end{aligned}$$

Remark 5.4. Theorem 5.3 and its proof below using odd reflections remain to be valid over an algebraically closed field of characteristic 0, where $j(\mu)$ (resp. $\mathtt{M}(\mu)$) for a partition μ is simply replaced by $\ell(\mu)$ (resp. the conjugate partition μ^t). In

this case, a classification of the finite-dimensional simple $spo(2n|\ell)$-modules using the Dynkin labels appeared in [9], whose sketchy proof uses totally different ideas and does not apply to the modular case. It can be shown that the statement in *loc. cit.* is equivalent to ours in characteristic zero.

The idea of using odd reflections for determining dominant weights (in characteristic zero) goes back at least to [11] (also see [14] for a general setup).

Remark 5.5. The simple *polynomial* $GL(n|m)$-modules (i.e. the composition factors appearing in various tensor powers of the natural $GL(n|m)$-module) were classified by Brundan-Kujawa [4] (this generalizes some earlier partial result of Donkin). According to Theorem 5.3, the simple $SpO(2n|2m+1)$-modules happen to admit the same parameterizing set as the simple polynomial $GL(n|m)$-modules. This remarkable fact suggests that the inclusion $GL(n|m) \leq SpO(2n|2m+1)$ could be significant for further development.

We will present a detailed proof below in the case of $G = SpO(2n|2m+1)$. Denoting the explicit set of weights in part (1) of the above theorem by $\mathcal{X}^\dagger(T)$ and recalling (4.1), we shall prove

$$X^\dagger(T) = \mathcal{X}^\dagger(T).$$

5.4. **The inclusion $\mathcal{X}^\dagger(T) \subseteq X^\dagger(T)$.** Let $\mathcal{G} = SpO(2N|2M+1)$ with $N \geq n$ and $M \geq m$. We denote by δ_i with $i \in I(N|M)$ the standard weights for $\mathcal{G}$, and by $T_{N,M}$ the maximal torus for $\mathcal{G}$ to distinguish from T for $G = SpO(2n|2m+1)$. Denote by S_{N+M} the symmetric group on the set $I(N|M)$ with subgroups S_N and S_M whose notations are self-explained. According to [9], up to the Weyl group equivalence, the systems of simple roots Π^σ are determined by the following set of minimal length coset representatives in $S_{N+M}/S_N \times S_M$:

$$D_{N,M} := \{\sigma \in S_{N+M} \mid \sigma^{-1}(-N) < \cdots < \sigma^{-1}(-1), \sigma^{-1}(1) < \cdots < \sigma^{-1}(M)\}.$$

More precisely, if we denote $\sigma(\delta_i) = \delta_{\sigma(i)}$ and $\sigma(\delta_i \pm \delta_j) = \delta_{\sigma(i)} \pm \delta_{\sigma(j)}$, etc, then $\Pi^\sigma := \{\sigma(x) \mid x \in \Pi\}$. Denote by $\Delta^{\sigma+}$ the set of positive roots relative to Π^σ. We shall denote by $L^\sigma(\lambda)$ the irreducible highest weight $Dist(\mathcal{G})$-module of highest weight λ relative to Π^σ. In particular, $L^1(\lambda) = L(\lambda)$.

Lemma 5.6. *A weight λ of the form $\lambda = \sum_{i=-N}^{-1} \lambda_i \delta_i$, where $(\lambda_{-N}, \ldots, \lambda_{-1})$ forms a partition, always belongs to $X^\dagger(T_{N,M})$.*

Proof. For $1 \leq r \leq N$, the r-th exterior power $\Lambda^r V$ of the natural $\mathcal{G}$-module $V = k^{2N|2M+1}$ is of highest weight $\Lambda_r := \sum_{i=-N}^{r-N-1} \delta_i$, and thus $\Lambda_r \in X^\dagger(T_{N,M})$. Denote the conjugate partition of $(\lambda_{-N}, \ldots, \lambda_{-1})$ by $(r_1, \ldots, r_s)$, with $N \geq r_1 \geq \cdots \geq r_s \geq 0$. Clearly $\lambda = \Lambda_{r_1} + \cdots + \Lambda_{r_s}$, and thus $L(\Lambda_{r_1}) \otimes \cdots \otimes L(\Lambda_{r_s})$ contains $L(\lambda)$ as a quotient $\mathcal{G}$-module. $\square$

Each Π^σ gives rise to a Borel subgroup B^σ of $\mathcal{G}$. The Verma module $M^\sigma(\lambda) := \mathrm{Dist}(\mathcal{G}) \otimes_{\mathrm{Dist}(B^\sigma)} k_\lambda$ has a unique irreducible quotient $\mathrm{Dist}(\mathcal{G})$-module $L^\sigma(\lambda)$. The method of *odd reflections* as presented in the next lemma is an effective tool of relating different conjugacy classes of Borel subgroups. It was exploited earlier by

Serganova and many other authors in characteristic zero and then formulated in [4, Lemma 4.2] for $GL(m|n)$ in characteristic $p > 2$. The following argument is adapted from *loc. cit.* in terms of roots and root vectors.

Lemma 5.7. *Let $\lambda \in X(T_{N,M})$, and let $\alpha = \pm\delta_i \pm \delta_j$ be an odd root. Suppose that $\sigma, \sigma' \in D_{N,M}$ are such that $\Delta^{\sigma'+} = \Delta^{\sigma+} \cup \{-\alpha\}\backslash\{\alpha\}$. Then,*

$$L^\sigma(\lambda) \cong \begin{cases} L^{\sigma'}(\lambda) & \text{if } (\lambda,\alpha) \equiv 0 \mod p, \\ L^{\sigma'}(\lambda-\alpha) & \text{if } (\lambda,\alpha) \not\equiv 0 \mod p. \end{cases}$$

Proof. Let v be a σ-highest weight vector in $L^\sigma(\lambda)$ of weight λ. For any $\beta \in \Delta^{\sigma+} \cap \Delta^{\sigma'+}$, we have

$$X_\beta X_{-\alpha} v = [X_\beta, X_{-\alpha}]v = 0 \tag{5.2}$$

since either $\beta - \alpha$ is not a root or it belongs to $\Delta^{\sigma+} \cap \Delta^{\sigma'+}$.

Since 2α is not a root and α is odd, $X_{-\alpha}^2 v = 0$.

If $X_{-\alpha}v = 0$, then v is a σ'-highest weight vector of weight λ, and $L^\sigma(\lambda) \cong L^{\sigma'}(\lambda)$.

If $X_{-\alpha}v \neq 0$, then $X_{-\alpha}v$ is a σ'-highest weight vector of weight $\lambda - \alpha$, and $L^\sigma(\lambda) \cong L^{\sigma'}(\lambda - \alpha)$.

Now, $X_{-\alpha}v \neq 0$ if and only if there exists $b \in Dist(B^\sigma)$ such that $bX_{-\alpha}v = v$. By (5.2), we only need to consider b to be a nonzero scalar multiple of X_α. Finally $X_\alpha X_{-\alpha} v = (\lambda, \alpha)v$. This completes the proof. □

Let $\mathbf{w} \in D_{N,M}$ be the distinguished permutation on $I(N|M)$ such that the sequence $\{\mathbf{w}(i)\}_{i\in I(N|M)}$ is the sequence which starts first from $-N$ to $n-N-1$ increasingly, then from 1 to M, and finally from $n-N$ to -1 increasingly. Recall the Mullineux map $\mathtt{M}$ from Proposition 5.2.

Proposition 5.8. *Let $N \geq n$ and $(\mu_{-N}, \mu_{1-N}, \dots, \mu_{-1})$ be a partition of length $\leq N$ such that the 'tail' $\mu^{>n} := (\mu_{n-N}, \mu_{n-N+1}, \dots, \mu_{-1})$ is restricted. Let $M \geq m$ be such that $M \geq \ell(\mathtt{M}(\mu^{>n}))$. Set $\mu = \sum_{i=-N}^{-1} \mu_i\delta_i$. Then we have an isomorphism of $\mathcal{G}$-modules $L(\mu) \cong L^{\mathbf{w}}(\mu^{\mathbf{w}})$ with*

$$\mu^{\mathbf{w}} := \sum_{i=-N}^{n-N-1} \mu_i\delta_i + \sum_{i=1}^{M} \mathtt{M}(\mu^{>n})_i\delta_i$$

where $\mathtt{M}(\mu^{>n})_i$ is the i-th part of the partition $\mathtt{M}(\mu^{>n}) := (\mathtt{M}(\mu^{>n})_1, \mathtt{M}(\mu^{>n})_2, \cdots)$.

Proof. We will apply Lemma 5.7 repeatedly with an ordered sequence of odd reflections associated to the following $M(N-n)$ odd roots:

$$\underbrace{\delta_{-1}-\delta_1, \delta_{-2}-\delta_1, \cdots, \delta_{n-N}-\delta_1}; \ \underbrace{\delta_{-1}-\delta_2, \delta_{-2}-\delta_2, \cdots, \delta_{n-N}-\delta_2}; \ \cdots ;$$
$$\underbrace{\delta_{-1}-\delta_M, \delta_{-2}-\delta_M, \cdots, \delta_{n-N}-\delta_M}. \tag{5.3}$$

In this way, Δ^+ is replaced by $\Delta^{\mathbf{w}+}$. After the first $N-n$ steps of odd reflections, the weight μ is replaced by

$$\tilde{\mu} := \sum_{i=-N}^{-1} \mu_i\delta_i - \sum_{i=n-N}^{-1} x_i\delta_i + \left(\sum_{i=n-N}^{-1} x_i\right)\delta_1.$$

Here the numbers $x_i \in \{0,1\}$ with $n-N \leq i \leq -1$ are defined starting from $i=-1$ by the following formula:

$$x_i = \begin{cases} 0 & \text{if } \mu_i + \sum_{a=i+1}^{-1} x_a \equiv 0 \mod p, \\ 1 & \text{if } \mu_i + \sum_{a=i+1}^{-1} x_a \not\equiv 0 \mod p. \end{cases}$$

By [4, Lemma 6.2], one has

$$\mu_i - x_i = J(\mu^{>n})_i, \text{ for } n-N \leq i \leq -1,$$

and thus

$$\tilde{\mu} = \sum_{i=-N}^{n-N-1} \mu_i \delta_i + \sum_{i=n-N}^{-1} J(\mu^{>n})_i \delta_i + j(\mu^{>n})\delta_1.$$

Here and below we write for $r \geq 1$ that $J^r(\mu^{>n}) = (J^r(\mu^{>n})_{n-N}, \cdots, J^r(\mu^{>n})_{-1})$.

Repeating the above reasoning for the next $N-n$ odd reflections among (5.3), we replace $\tilde{\mu}$ by the weight

$$\sum_{i=-N}^{n-N-1} \mu_i \delta_i + \sum_{i=n-N}^{-1} J^2(\mu^{>n})_i \delta_i + j(\mu^{>n})\delta_1 + j(J(\mu^{>n}))\delta_2$$

and so on. Recall the definition of the Mullineux map $\mathtt{M}$ and note that $J^M(\mu^{>n}) = \emptyset$ thanks to $M \geq \ell(\mathtt{M}(\mu^{>n}))$. Finally, after all the $M(N-n)$ odd reflections, we end up with the weight $\mu^{\mathbf{w}}$ in the proposition. $\square$

Now we are ready to prove that $\mathfrak{X}^\dagger(T) \subseteq X^\dagger(T)$ for $SpO(2n|2m+1)$. First, take $\nu = \sum_{i \in I(n|m)} \nu_i \delta_i \in \mathfrak{X}^\dagger(T)$ with the additional assumption that $\nu^+ := (\nu_1, \ldots, \nu_m)$ is restricted. Denote by $\mathtt{M}(\nu^+) := (a_1, a_2, \ldots)$ which is also a restricted partition. It follows by the definition of $\mathfrak{X}^\dagger(T)$ that $(\nu_{-n}, \ldots, \nu_{-1}, a_1, a_2, \ldots)$ is a partition. Choose $M = m$ and $N \geq n + \ell(\mathtt{M}(\nu^+))$. By Lemma 5.6, the weight

$$\mu := \sum_{i=-N}^{n-N-1} \nu_{i+N-n} \delta_i + \sum_{i=n-N}^{-1} a_{i+N-n+1} \delta_i$$

lies in $X^\dagger(T_{N,M})$. Note that $\mathbf{w}$ has been chosen such that the standard set of positive roots for $SpO(2n|2m+1)$ is a subset of positive roots relative to $\pi^{\mathbf{w}}$ for $\mathcal{G} = SpO(2N|2M+1)$. This inclusion of roots is compatible with the non-standard inclusion $I(n|m) \hookrightarrow I(N|M)$ given by $(0>)i \mapsto i+n-N, (0<)j \mapsto j$, and it gives rises to an embedding of supergroups $SpO(2n|2m+1) \leq \mathcal{G}$. Applying Proposition 5.8 and $\mathtt{M}^{-1} = \mathtt{M}$, we have $\mu^{\mathbf{w}}|_T = \nu$. We conclude that $\nu \in X^\dagger(T)$ by restricting the $\mathcal{G}$-module $L^{\mathbf{w}}(\mu^{\mathbf{w}})$ to the subgroup $SpO(2n|2m+1)$.

Write an arbitrary $\lambda \in \mathfrak{X}^\dagger(T)$ (uniquely) as $\lambda = \nu + p\sum_{i=1}^m \tau_i \delta_i$ for some partition $\tau = (\tau_1, \ldots, \tau_m)$ and $\nu \in \mathfrak{X}^\dagger(T)$ with ν^+ being restricted. Since $j(\nu^+) = j(\lambda^+)$ by (5.1), we have $\nu \in \mathfrak{X}^\dagger(T)$. By the tensor product Theorem 4.5,

$$L(\lambda) \cong L(\nu) \otimes L_{\text{ev}}(\sum_{i=1}^m \tau_i \delta_i)^{[1]},$$

whence $\lambda \in X^\dagger(T)$ and $\mathfrak{X}^\dagger(T) \subseteq X^\dagger(T)$.

5.5. **The inverse inclusion $X^\dagger(T) \subseteq \mathfrak{X}^\dagger(T)$.** Part of the necessary conditions for $\lambda \in X^\dagger(T)$ is easy to determine.

Lemma 5.9. *Assume $\lambda = \sum_{i\in I(n|m)} \lambda_i\delta_i \in X^\dagger(T)$. Then, $(\lambda_{-n}, \dots, \lambda_{-1})$ and $\lambda^+ := (\lambda_1, \dots, \lambda_m)$ are partitions.*

Proof. Since Δ^+ contains $\Delta^+(Sp(2n))$, the fact that λ belongs to $X^\dagger(T)$ implies that $(\lambda_{-n}, \dots, \lambda_1)$ is a partition by the classical result (cf. e.g. [8]). Similarly, Since Δ^+ contains $\Delta^+(SO(2m+1))$, λ^+ is also a partition. □

To prove $X^\dagger(T) \subseteq \mathfrak{X}^\dagger(T)$, it remains to show that $j(\lambda^+) \le \lambda_{-1}$ for $\lambda \in X^\dagger(T)$. To that end, we first consider the special case of $SpO(2|2m+1)$ (i.e. $n = 1$) whose maximal torus will be denoted by $T_{1,m}$. Recall the standard set of simple roots for $SpO(2|2m+1)$ is $\Pi = \{\delta_{-1}-\delta_1, \delta_1-\delta_2, \dots, \delta_{m-1}-\delta_m, \delta_m\}$, with $\delta_{-1}-\delta_1$ being odd. Let $\mathbf{z} \in S_{1+m}$ be the permutation defined by $\mathbf{z}(-1) = 1, \mathbf{z}(1) = 2, \cdots, \mathbf{z}(m-1) = m, \mathbf{z}(m) = -1$. Then

$$\Pi^{\mathbf{z}} = \{\delta_1 - \delta_2, \dots, \delta_{m-1} - \delta_m, \delta_m - \delta_{-1}, \delta_{-1}\},$$

with $\delta_m - \delta_{-1}$ and δ_{-1} being odd.

Via an ordered sequence of odd reflections associated to the m odd roots

$$\delta_{-1} - \delta_1, \delta_{-1} - \delta_2, \dots, \delta_{-1} - \delta_m,$$

the highest weight λ relative to the standard Borel is replaced by some $\lambda^{\mathbf{z}} = \sum_{i\in I(1|m)} \lambda^{\mathbf{z}}_i\delta_i$, with all $\lambda^{\mathbf{z}}_i \in \mathbb{Z}$, which is a highest weight relative to the Borel $B^{\mathbf{z}}$. That is, $L(\lambda) \cong L^{\mathbf{z}}(\lambda^{\mathbf{z}})$. Note that $\lambda^{\mathbf{z}+} := (\lambda^{\mathbf{z}}_1, \dots, \lambda^{\mathbf{z}}_m)$ is a partition by considering the restriction of $L^{\mathbf{z}}(\lambda^{\mathbf{z}})$ to the subgroup $SO(2m+1)$ since the standard Borel of $SO(2m+1)$ is a subgroup of the Borel $B^{\mathbf{z}}$ of $SpO(2|2m+1)$.

Lemma 5.10. *Assume $\lambda = \sum_{i\in I(1|m)} \lambda_i\delta_i \in X^\dagger(T_{1,m})$. Retain the notations as above. Then, we have*

$$\begin{aligned} \lambda^{\mathbf{z}}_{-1} &\in \mathbb{Z}_+ \\ J(\lambda^{\mathbf{z}+}) &= \lambda^+ \\ j(\lambda^{\mathbf{z}+}) &= \lambda_{-1} - \lambda^{\mathbf{z}}_{-1} \\ j(\lambda^+) &\le \lambda_{-1}. \end{aligned} \tag{5.4}$$

Proof. We have $\lambda^{\mathbf{z}}_{-1} \in \mathbb{Z}_+$ since $2\delta_{-1}$ is a positive root in $\Pi^{\mathbf{z}}$.

Via an ordered sequence of odd reflections associated to the m odd roots

$$\delta_m - \delta_{-1}, \dots, \delta_1 - \delta_{-1},$$

the weight $\lambda^{\mathbf{z}}$ relative to the Borel $B^{\mathbf{z}}$ is replaced by the weight λ. The identity $J(\lambda^{\mathbf{z}+}) = \lambda^+$ now follows by an argument which is completely parallel to the one for Proposition 5.8 (also compare the proof of [4, Lemma 6.2]).

Next, we have

$$\begin{aligned} j(\lambda^{\mathbf{z}+}) &= |\lambda^{\mathbf{z}+}| - |J(\lambda^{\mathbf{z}+})| \\ &= |\lambda^{\mathbf{z}+}| - |\lambda^+| = \lambda_{-1} - \lambda^{\mathbf{z}}_{-1} \end{aligned}$$

where the last equation is a byproduct of the above procedure of odd reflections.

Finally, (5.4) follows from a direct computation:

$$\begin{aligned} j(\lambda^+) &= j(J(\lambda^{\mathbf{z}+})) \\ &\le j(\lambda^{\mathbf{z}+}) = \lambda_{-1} - \lambda^{\mathbf{z}}_{-1} \le \lambda_{-1}. \end{aligned}$$

□

The supergroup $SpO(2|2m+1)$ can be regarded naturally as a subgroup of $SpO(2n|2m+1)$ in a way compatible with the natural inclusion $I(1|m) \subseteq I(n|m)$, and thus the corresponding fundamental systems of the two supergroups are compatible. Hence, $\lambda = \sum_{i \in I(n|m)} \lambda_i \delta_i \in X^\dagger(T)$ implies that $\sum_{i \in I(1|m)} \lambda_i \delta_i \in X^\dagger(T_{1,m})$, by restricting the $SpO(2n|2m+1)$-module $L(\lambda)$ to the subgroup $SpO(2|2m+1)$. It follows by (5.4) that $j(\lambda^+) \le \lambda_{-1}$ for $\lambda \in X^\dagger(T)$. This together with Lemma 5.9 imply that $X^\dagger(T) \subseteq \mathfrak{X}^\dagger(T)$ in the general case of $SpO(2n|2m+1)$. The proof of Theorem 5.3 for $SpO(2n|2m+1)$ is now completed.

Remark 5.11. Theorem 5.3 for the classification of simple $SpO(2n|2m)$-modules can be established using the same ideas in two steps as above and so we will skip the details. The only difference here is that the odd reflections associated to the odd roots $\delta_i + \delta_j$ will also be used. The appearance of the absolute value $|\lambda_{n+m}|$ comes from the weight condition of $SO(2m)$ (compare Lemma 5.9 and its proof).

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